SIMPLE GROUPS HAVING p-NILPOTENT 2nd-MAXIMAL SUBGROUPS

BY

AVINOAM MANN

ABSTRACT

Theorem. Let G be a finite simple group. Assume (i) $3 \,\|\, |G|$. *(ii) Each 2nd-maximal subgroup of G has a normal* 3-*complement. Then* $G \simeq PSL(2,q)$, *for some q.*

Part of the argument is isolated to give a non-existence theorem for simple groups with a special 3-Sylow structure. Generalizations are discussed.

1. The well-known Schmidt-Iwasawa theorem states the following: *If each proper subgroup of the finite group G is nilpotent, then G is solvable.*

Among the generalization of this beautiful result, let us mention the following two:

If each proper subgroup of the finite group G is p-nilpotent, then G is solvable or p-nilpotent [15, Propositions 1, 2].

If each $2nd$-maximal subgroup of the finite group G is nilpotent, and G is not solvable, then $G \cong A_5$ or $G \cong SL(2,5)$ [19, 16].

(p is a fixed prime; a 2nd-maximal subgroup is one which is maximal in a maximal subgroup; notice that the hypothesis of the Schmidt-Iwasawa theorem is equivalent to "each maximal subgroup is nilpotent").

These two results suggest naturally the investigation of finite groups, whose 2nd-maximal subgroups are p-nilpotent. The structure of simple groups of this type, in the case $p = 2$, was studied in the author's thesis [17]. His results, however, are included in a much more general theorem of J. G. Thompson [20]. The structure of the non-simple and non-solvable groups of this type was determined by Berkovitch [1].

In this paper we are interested in the case $p = 3$, and prove the following:

THEOREM 1. *Let G be finite simple non-abelian group. If the order of G is divisible by* 3, *and if each* 2*nd-maximal subgroup of G is* 3-*nilpotent, then* $G \cong PSL(2,q)$, *for some q.*

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This result is proved in Section 4. In Sections 2 and 3 we obtain some results on the structure of G without assuming $p = 3$. Among other results, it is shown that a Sylow p-subgroup, P, is either elementary abelian or extra-special, and is usually a (TI)-set. When $p = 3$, results of Feit-Thompson [6] and Herzog [12] are used to dispose of the cases where P is cyclic or abelian non-cyclic, respectively(1). When P is extra-special, a character theoretic argument yields a contradiction. This argument can be isolated, and is stated in Theorem 2 as a non-existence theorem for simple groups with a rather special 3-Sylow structure. This is considered in Section 5, where we also consider the possibility of extending our results to all p.

Notation and terminology. G denotes always a finite group. If X is a subset of G, $|X|$ is the number of elements in X. A subgroup H is a *p-complement*, if $|H|$ is prime to p and the index $|G:H|$ is a power of p. G is *p-nilpotent*, if it has a normal p-complement. An *S group* is a non-nilpotent group, each proper subgroup of which is nilpotent. $Z(G)$ and G' denote respectively, the center and commutator subgroup of G. An *extra-special* p-group is a non-abelian p-group, in which $G/Z(G)$ is elementary abelian, $Z(G) = G'$ and $|Z(G)| = p$. A p'-group is a group whose order is prime to p. If X is a subset, and H a subgroup, of G, then $C_H(X)$ and $N_H(X)$ are the centralizer and normalizer, respectively, of X in H. We shall also write $C_G(X) = C(X)$, $N_G(X) = N(X)$. X is a (TI) set in G, if, for any $g \notin N(X)$, $X \cap X^g$ is empty or contains only the identity.

We refer the reader to [10, Section 14.4] for the definition and properties of p-normality.

The following (well-known) facts on the structure of S groups will be repeatedly used: An S group, G, has order $p^{\alpha}q^{\beta}$, where p and q are distinct primes. Denote by P a p-Sylow subgroup of G and by Q a q-Sylow subgroup. Then one Sylow subgroup, P say, is normal in G, and then Q is cyclic. P/P' and P' are both elementary abelian, and if $P' \neq 1$, then $P' = Z(P)$. Q acts irreducibly on P/P'. If Q_1 is the subgroup of index q in Q, then $Z(G) = Q_1 \times P'$. If $|P:P'| = p^m$, then m is the order of p (mod. q). If p is odd, then the exponent of P is p.

All groups in this paper are finite.

2. We begin with the case of a general p. Throughout the paper, p denotes some fixed prime, G is a finite simple group, and we assume

a. $p \mid |G|$,

b. Each 2nd-maximal subgroup of G is p-nilpotent.

LEMMA 1. *Each proper subgroup of G is either p-nilpotent or an S group. Any S subgroup of G is a maximal subgroup.*

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This follows immediately from our assumptions and [15, Propositions 1, 2].

LEMMA 2. *G is p-normal.*

Proof. Suppose not. By a result of Burnside [10, 4.2.5–14.4.3], there exists a p-subgroup, Q, of G, which is not a Sylow subgroup, and such that $N(Q)/C(Q)$ is not a p-group. In particular, $N(Q)$ is not p-nilpotent. By Lemma 1, $N(Q)$ is an S group. Moreover, the Sylow p-subgroup, P, of $N(Q)$, is normal in $N(Q)$, otherwise $N(Q)$ would be p-nilpotent. Since Q is not a Sylow subgroup of G, it is also not a Sylow subgroup of $N(Q)$: $Q \neq P$. Since $Q \lhd N(Q)$, $Q \subseteq P$ and $Q \neq P$, the structure of S groups shows that $Q \subseteq Z(N(Q))$, $N(Q) = C(Q)$, and $N(Q)/C(Q) = 1$, a contradiction.

Let P be a fixed p-Sylow subgroup of G, Z — the center of P, and $M = N(Z)$.

LEMMA 3. *M is an S group and a maximal subgroup of G. Also, $M = N(P)$.*

Proof. The simplicity and p-normality of G imply by the Hall-Grün theorem [10, 14.4.6], that M is not p-nilpotent. Hence M is an S group possessing a normal p-Sylow subgroup. Therefore M is maximal. Since $P \subseteq M$, P is a p-Sylow subgroup of M, so $P \lhd M$. By maximality of M, $M = N(P)$.

LEMMA 4. *Any proper subgroup of G which is not p-nilpotent is conjugate to M.*

Proof. Let K be any non-p-nilpotent proper subgroup of G. Then K must be a maximal subgroup, and K is an S group with a normal p-Sylow subgroup. Let K_p be this Sylow subgroup. Then $K = N(K_p)$ follows from the maximality of K and simplicity of G. Hence K_p is a Sylow subgroup of G. Therefore K_p is conjugate to P and K is conjugate to M.

As an S group, M has order $p^n q^m$, for some prime q, $q \neq p$. Also, if Q denotes a q-Sylow subgroup of M, Q is cyclic.

LEMMA 5. $m = 1$.

Proof. Suppose $m > 1$. Let T be any non-identity proper subgroup of Q. Then $T \lhd M$, by the structure of M, so maximality of M implies $M = N(T)$. T is characteristic in the cyclic group Q, so $T \lhd N(Q)$, and $N(Q) \subseteq N(T) = M$. Therefore Q is a Sylow subgroup of $N(Q)$, hence also of G. For each non-identity subgroup S of Q, we have shown that $N(S) \subseteq M$, and therefore $N(S)$ is q-nilpotent. Frobenius' theorem [10, 14.4.7] shows that G has a normal q-complement, and is not simple, a contradiction.

If A is a p-subgroup of G, and r is a prime, $И(A; r)$ denotes the set of r-subgroups of G that are normalized by A.

Now we quote the following result.

TRANSITIVITY THEOREM. *Let H be a simple group such that each proper subgroup of G is p-solvable, for some (fixed) prime p. Let P be a p-Sylow subgroup of H, A a maximal normal abelian subgroup of P, and $q \neq p$ a prime. If A cannot be generated by less than three elements, then any two maximal elements of $\mathcal{U}(A;q)$ are conjugate under an element of $C_H(A)$.*

This is proved in [7, Th. 17.1], assuming that each proper subgroup of H is solvable. It is known that the theorem holds also under the assumption of p-solvability of proper subgroups, or even weaker conditions. A discussion of this may be found in a forthcoming book of D. Gorenstein [9].

By Lemma 1, our group G and the given prime p satisfy the assumptions of the transitivity theorem.

LEMMA 6. *P is the unique Sylow p-subgroup of G containing Z.*

Proof. By definition, $M = N(Z)$. Let P_1 be any Sylow subgroup containing Z. By p-normality, $Z = Z(P_1)$. Hence $Z \lhd P_1$, so $P_1 \subseteq N(Z) = M$. As P is the unique Sylow p-subgroup of M, $P_1 = P$.

LEMMA 7. *Let A be a maximal normal abelian subgroup of P. Then $A = C(A)$.*

Proof. Obviously, $A \supseteq Z$. Therefore $A \lhd P$. If P_1 is a Sylow p-subgroup of $N(A)$, then $P_1 \supseteq A \supseteq Z$ and the previous lemma imply $P_1 = P$. Therefore P is the unique Sylow p-subgroup of $N(A)$, so $P \lhd N(A)$ and $N(A) \subseteq N(P) = M$. Hence $C(A) = C_M(A) = A$.

LEMMA 8. *Let A be a maximal normal abelian subgroup of P. If A cannot be generated by less than three elements, then $\mathcal{U}(A;r) = 1$, for any prime $r \neq p$.*

Proof. Let R and R_1 be two maximal elements of $\mathcal{U}(A;r)$. By the transitivity theorem, $R_1 = R^a$, with $a \in C(A)$. By Lemma 7, $a \in A$. Since $A \subseteq N(R)$, $R_1 = R$, and R is the unique maximal element of $\mathcal{U}(A;r)$.

Let $g \in P$. Then $A^g = A$, therefore g transforms $\mathcal{U}(A;r)$ onto itself. In particular, we must have $R^g = R$ for R the unique maximal element in $\mathcal{U}(A;r)$. Hence $P \subseteq N(R)$ and $R \in \mathcal{U}(P;r)$. Let $R_1 \in \mathcal{U}(P;r)$. Then $R_1 \in \mathcal{U}(A;r)$, therefore R_1 is contained in the unique maximal element of $\mathcal{U}(A;r)$, $R_1 \subseteq R$. Therefore R is also the unique maximal element of $\mathcal{U}(P;r)$. Now repetition of the argument showing $P \subseteq N(R)$ yields $M = N(P) \subseteq N(R)$. Therefore $R \lhd MR$. As M is maximal, and G is simple, it follows that $M = MR$ and $R \lhd M$. Since M has no normal p'-subgroup, $R = 1$.

LEMMA 9. *Suppose P is abelian, and $|P| \geqq p^3$. Then, for any $1 \neq a \in P$, $C(a) = P$.*

Proof. Obviously, $P \subseteq C(a)$. The structure of M is such, that if P is abelian, then it is elementary abelian. Therefore $|P| \geqq p^3$ implies that P cannot have less than three generators. Also, $Z(M) = 1$. Therefore, $C(a)$ is not conjugate to M, as $a \in Z(C(a))$. By Lemma 4, $C(a)$ has a normal p-complement, T say. Therefore P normalizes T, and $(|P|, |T|) = 1$. It is well-known that in these circumstances P normalizes some r-Sylow subgroup, R, of T, for any prime divisor r of $|T|$. Now Lemma 8 implies $R = 1$, and therefore also $T = 1$, so $C(a) = P$.

Now consider the case in which P is non-abelian. Denoting again $Z = Z(P)$, both Z and P/Z are elementary abelian.

LEMMA 10. *If P is non-abelian, it is an extra-special group.*

Proof. We have remarked that P/Z and Z are elementary abelian. Hence we need prove only that $|Z| = p$.

Let Q be a q-Sylow subgroup of M. Then $Z = Z(M)$ implies $Z \subseteq N(Q)$. Suppose $N(Q) = QZ$. This implies, first, that Q is a Sylow subgroup of G, and, second, that Q is in the center of $N(Q)$. Hence G has a normal q-complement [10, 14.3.1], and is not simple. Therefore $N(Q) \neq QZ$.

Let P_1 be a p-Sylow subgroup of $N(Q)$ containing Z. By Lemma 6, $P_1 \subseteq P$. Hence $P_1 = N_P(Q) = Z$. Therefore $N(Q)$ cannot be conjugate to M, so, by Lemma 4, $N(Q)$ has a normal p-complement, T say. Now $N(Q) = TZ \neq QZ$, so $T \neq Q$.

Z acts on the group T/Q. Let $1 \neq z \in Z$. Then $z \in Z(M)$, so $M = C(z)$. Therefore $N(Q) \cap C(z) = QZ$, so $C_T(z) = Q$. As $(|T|, p) = 1$, this implies $C_{T/Q}(z) = 1$ [8, Th. 1]. Hence Z acts as a group of fixed-point-free automorphisms on T/Q. According to Burnside [3, p. 335], Z is either cyclic or a generalized quaternion group. As Z is elementary abelian, we must have $|Z| = p$.

LEMMA 11. *Suppose P is extra-special, $|P| > p^3$, and p is odd. Then for any $a \in P - Z$, $C(a) \subseteq P$.*

Proof. Since P/Z is abelian, we have $\langle a, Z \rangle \lhd P$. Therefore all the conjugates of a in P are contained in $\langle a, Z \rangle$. P has exponent p, because M is an S group and p is odd. Therefore $|\langle a, Z \rangle| = p^2$, so a has less than p^2 conjugates in P. Since $a \notin Z$, we must have $|P : C_P(a)| = p$. Since $|P| > p^3$, there exists an element b, $b \in C_P(a) - \langle a, Z \rangle$. The group $\langle a, b, Z \rangle$ is then elementary abelian of order p^3. Let A be a maximal normal abelian subgroup of P containing $\langle a, b, Z \rangle$. Then A does not have less than three generators.

Since $a \notin Z$, $C(a) \neq M$. Suppose $C(a)$ is conjugate to M, and let P_1 be the (unique) Sylow p-subgroup of $C(a)$. Then $P_1 \neq P$. Since $Z(M) = Z$, and $|Z| = p$, we find $Z(P_1) = Z(C(a)) = \langle a \rangle$. Therefore $Z(P_1) \subseteq P$, which contradicts Lemma 6. Therefore $C(a)$ is not conjugate to M.

Now Lemma 4 implies that $C(a)$ has a normal p-complement, T say. Since $A \subseteq C(a)$, $T = 1$ follows as in the proof of Lemma 9. Hence $C(a)$ is a p-group. However, $Z \subseteq C(a)$, so, by Lemma 6 again, $C(a) \subseteq P$.

LEMMA 12. *If P is as in Lemma 11, P is a (TI) set.*

Proof. Let P_1 be any other Sylow p-subgroup of G, and suppose $a \in P \cap P_1$, $a \neq 1$. By Lemma 6, $a \notin Z$. Hence $Z(P_1) \subseteq C(a) \subseteq P$, by the preceding lemma, a contradiction.

3. We are now going to develop the necessary facts on the characters of G. We assume, in addition to the assumptions of Section 2, that p is odd, P is an extra-special group, and $|P| > p^3$.

The above assumptions imply that the order of P is p^{2n+1} for some natural number n, $n \geqq 2$. Denoting again $|M:P| = q$, we know that q is prime (Lemma 5), and, M being an S group, that $2n$ is the order of p (mod. q). Therefore $q \mid p^{2n} - 1$, but $q \nmid p^n - 1$, so $q \mid p^n + 1$. Hence q is odd. Since $p^n + 1$ is even, $q \leqq \frac{1}{2}(p^n + 1)$. Define t by

$$t = \frac{p^{2n} - 1}{q} \tag{1}$$

then $t = (p^n - 1)\dfrac{p^n + 1}{q}$, so $t \geqq 2(p^n - 1)$. Noting that $p^n \geqq 9$, we obtain

$$t > 3q. \tag{2}$$

We begin by considering the characters of P. First, P has p^{2n} linear characters, which we will denote by $\zeta_0, \zeta_1, \cdots, \zeta_{p^{2n}-1}$, with $\zeta_0 = 1$. By [11, p. 17], P has also $p-1$ characters of degree p^n. These will be denoted by $\eta_1, \cdots, \eta_{p-1}$. Since $p^{2n} + (p-1)(p^n)^2 = p^{2n+1}$, we have exhausted the characters of P.

Let $a \in P - Z$. We have seen in the proof of Lemma 11, that $|P:C_P(a)| = p$, hence $|C_P(a)| = p^{2n}$. The orthogonality relations of characters imply

$$\sum |\zeta_i(a)|^2 + \sum |\eta_i(a)|^2 = p^{2n}$$

However, $|\zeta_i(a)| = 1$ for each i, and there are p^{2n} ζ_i's. Therefore $\sum |\eta_i(a)|^2 = 0$, implying

$$\eta_i(a) = 0, \quad a \in P - Z. \tag{3}$$

Now consider the characters of M. There are q linear ones, which we denote by $\mu_0, \cdots, \mu_{q-1}$, with $\mu_0 = 1$. Note that $\mu_{i|P} = \zeta_0$.

If α is a character of P, we denote by $\tilde{\alpha}$ the induced character on M. M/Z is a Frobenius group, with kernel P/Z. $\zeta_0, \cdots, \zeta_{p^{2n}-1}$ may be considered as characters of P/Z, so the character theory for Frobenius groups [e.g. 4, pp. 171–172] shows that among $\tilde{\zeta}_1, \cdots, \tilde{\zeta}_{p^{2n}-1}$ there are exactly t different characters, which we may assume to be $\tilde{\zeta}_1, \cdots, \tilde{\zeta}_t$, and these are irreducible.

Now consider $\tilde{\eta}_i$, for some i, $1 \leqq i \leqq p-1$. Let $m \in M$. Since Z is the center of M, and η_i vanishes outside Z, we find $\eta_i^m = \eta_i$. Therefore, if $m_1, \cdots, m_q$ are representatives of the cosets of P in M, $\tilde{\eta}_{i|P} = \eta_i^{m_1} + \cdots + \eta_i^{m_q} = q\eta_i$.

S nce $P \lhd M$, $\tilde{\eta}_i$ vanishes outside P. Therefore

$$\begin{aligned}(\tilde{\eta}_i, \tilde{\eta}_i)_M &= \frac{1}{|M|}\,|P|(\tilde{\eta}_{i|P}, \tilde{\eta}_{i|P})_P \\ &= \frac{1}{q}(q\eta_i, q\eta_i)_P \\ &= q\end{aligned}$$

and so $\tilde{\eta}_i$ is a sum of at most q irreducible characters of M.

Since Z is the center of M, the elements of Z constitute p conjugacy classes in M. If $a \in P - Z$, we already know that $|C_M(a)| = p^{2n}$, so a has pq conjugates in M. There are $p^{2n+1} - p$ elements in $P - Z$, so they constitute $1/q(p^{2n} - 1) = t$ classes in M. Next, if $m \in M - P$, then the order of m is divisible by q, so m is conjugate to an element of ZQ. Therefore $|C_M(m)| = pq$, and m has p^{2n} conjugates. As there are $p^{2n+1}q - p^{2n+1}$ such elements, they constitute $pq - p$ classes, yielding a total of $pq + t$ classes of M. This, then, is also the number of characters of M. In addition to the μ_i's and $\tilde{\zeta}_i$'s there must therefore be still $(p-1)q$ other irreducible characters of M.

Let λ be any of the missing characters. If $\lambda|P = \zeta_0$, then $P \leqq \ker \lambda$, so λ is a character of M/P, so is linear, $\lambda = \mu_i$ for some i. Next, if $\lambda|P$ involves some $\zeta_i\,(i \neq 0)$, then λ is a constituent of $\tilde{\zeta}_i$. Since $\tilde{\zeta}_i$ is irreducible, we get $\lambda = \tilde{\zeta}_i$. Therefore $\lambda|P$ must involve some η_i, so λ is a constituent of $\tilde{\eta}_i$. Since all $(p-1)q$ missing characters are constituents of at least one of the $p-1$ characters $\tilde{\eta}_1, \cdots, \tilde{\eta}_{p-1}$, and each $\tilde{\eta}_i$ is a sum of at most q irreducible characters, we find that, in fact, each $\tilde{\eta}_i$ is a sum of exactly q different irreducible characters, yielding altogether all of the $(p-1)q$ missing characters.

Let $\lambda_1, \cdots, \lambda_{(p-1)q}$ be the irreducible characters of M other than the μ_i's and $\tilde{\zeta}_i$'s. We have just seen that for each λ_i there exists exactly one η_j such that $(\lambda_{i|P}, \eta_j) = 1$, and $(\lambda_{i|P}, \eta_k) = 0$ for $k \neq j$. We have also seen that $(\lambda_{i|P}, \zeta_j) = 0$ for $j \neq 0$, while $(\lambda_{i|P}, \zeta_0) = 0$ follows from $\tilde{\zeta}_0 = \mu_0 + \mu_1 + \cdots + \mu_{q-1}$. Therefore $\lambda_{i|P} = \eta_j$, so, by (3)

$$\lambda_i(a) = 0, \quad a \in P - Z \tag{4}$$

We now come to the characters of G. By Lemma 12, P is a (TI) set. Also $M = N(P)$, and $\tilde{\zeta}_1, \cdots, \tilde{\zeta}_t$ are characters of M vanishing outside P. The theory of exceptional characters [e.g. 5] now implies that there are t irreducible characters of G, say $\chi_1, \cdots, \chi_t$, and a sign ε, such that

$$(\zeta_i - \zeta_j)^* = \varepsilon(\chi_i - \chi_j) \qquad i,j, = 1, \cdots, t \tag{5}$$

Here, if α is a class function of any subgroup of G, α^* denotes the induced class function of G.

Let $(\zeta_i^*, \chi_i) = \varepsilon + a_i$, for some integer a_i. Then (5) shows that, for $j \neq i$, $(\zeta_j^*, \chi_i) = a_i$. The lemma in [5] implies that $(\zeta_i^*, \varepsilon(\chi_i - \chi_j))_G = (\zeta_i^*, (\zeta_i - \zeta_j)^*)_G = (\tilde{\zeta}_i, \tilde{\zeta}_i - \tilde{\zeta}_j)_M = 1$, so $\varepsilon(\varepsilon + a_i - a_j) = 1$, and $a_i = a_j = a$ (say). Let λ be a character of M different from all the ζ_i's. Then $(\lambda, \zeta_i - \zeta_j) = 0$ implies as above, $(\lambda^*, \chi_i - \chi_j) = 0$. From the Frobenius reciprocity theorem we get

$$\chi_{i|M} = \varepsilon\zeta_i + a \sum_1^t \zeta_j + \sum_0^{q-1} b_j\mu_j + \sum_1^{(p-1)q} c_j\lambda_j$$

Taking into account (4), and the relations $\mu_{j|P} = \zeta_0$, $\sum_1^t \zeta_{j|P} = \sum_{j\neq 0}\zeta_j$, and $\sum_{j\neq 0}\zeta_j(a) = -1$, for $a \in P - Z$, we get

$$\chi_{i|P-Z} = \varepsilon\zeta_{i|P-Z} + c \tag{6}$$

where c is an integer which is independent of i.

Let $\theta_1, \cdots, \theta_s$ be the non-exceptional characters of G (i.e. those different from $\chi_1, \cdots, \chi_t$). Then $(\theta_k, \chi_i - \chi_j) = 0$ and the reciprocity theorem show that all the ζ_i's have the same multiplicity in $\theta_{k|M}$. It follows, in the same way as for (6), that there exists an integer c_k so that

$$\theta_{k|P-Z} = c_k \tag{7}$$

Let $a \in P - Z$. Then we have seen that $|C_G(a)| = |C_M(a)| = p^{2n}$. The orthogonality relations in M yield

$$p^{2n} = \sum_1^t |\zeta_i(a)|^2 + \sum_0^{q-1} |\mu_i(a)|^2 + \sum_1^{(p-1)q} |\lambda_i(a)|^2$$

in view of (4) and $\mu_{i|P} = \zeta_0$, this means

$$\sum_1^t |\zeta_i(a)|^2 = p^{2n} - q, \quad a \in P - Z \tag{8}$$

The orthogonality relations in G yield

$$\begin{aligned} p^{2n} &= \sum_1^t |\chi_i(a)|^2 + \sum_1^s |\theta_i(a)|^2 \\ &= \sum_1^t |\varepsilon\zeta_i(a) + c|^2 + \sum c_i^2 \\ &= \sum_1^t |\zeta_i(a)|^2 + 2\varepsilon c \sum_1^t \mathrm{Re}\,\zeta_i(a) + tc^2 + \sum c_i^2 \end{aligned}$$

using (8) and $\sum_1^t \zeta_i(a) = -1$, we obtain

$$p^{2n} = p^{2n} - q - 2\varepsilon c + tc^2 + \sum c_i^2$$

$$tc^2 \leqq q + 2\varepsilon c \tag{9}$$

Suppose $c \neq 0$. Then (9) yields $t \leqq q+2$, which contradicts (2). Therefore

$$c = 0 \tag{10}$$

It is also possible to show the existence of characters of G which play the role of exceptional characters with respect to the characters η_i of P. These characters can be constructed as a special case of the construction in Section 13 of [7]. However, these results are not needed for the proof of our theorem.

4. **Proof of the main theorem.** Let G be a simple group of order divisible by 3, such that each 2nd-maximal subgroup of G is 3-nilpotent. Let P, M, etc. have the same meaning as in Sections 2 and 3, with $p = 3$.

If $|P| = 3$, then, by a result of Feit-Thompson [6, Cor], G is isomorphic to either $PSL(2,5) \cong A_5$ or to $PSL(2,7)$.

If $|P| = 9$, then, since q is a prime dividing $|P| - 1$, $q = 2$. This is a contradiction, since the order of 3 (mod. 2) is 1 and not 2.

Hence, if P is abelian, we may assume $|P| \geqq 27$. This implies, as for $|P| = 9$, that $q \neq 2$. Therefore q is odd and $q \neq |P| - 1$. Now $G \cong PSL(2, |P|)$ follows from Lemma 9 and [12, Th. 5.1].

From now on we shall assume that P is not abelian. Eventually, this assumption will lead to a contradiction.

By Lemma 10, P is an extra-special group. If $|P| = 27$, then, since now $q \,\Big|\, \frac{1}{p}|P| - 1$, we get the same contradiction as for $|P| = 9$. Hence $|P| > 27$ and q is odd, so Lemmas 11, 12 and all the results of section 3 apply.

The key result here is the following.

LEMMA 13. *Let a, $b \in G$, $c \in P - Z$, $ab = c$ and $a^3 = b^3 = c^3 = 1$. Then, unless perhaps a and b are conjugate to each other and to c^{-1}, $a \in P$ and $b \in P$.*

Proof. Let $H = \langle a, b, c\rangle$. By [6, Th. 1] H contains a normal abelian subgroup, K say, such that $|H:K| = 3$. Suppose $3 \,\big|\, |K|$, and let L be the Sylow 3-subgroup of K. Then $L \lhd H$, so L is contained in all Sylow 3-subgroups of H. Letting P_1 be any Sylow 3-subgroup of G containing L, Lemma 12 implies that all Sylow 3-subgroups of H are contained in P_1, therefore $P_1 = P$ and $H \subseteq P$.

Suppose, then, that $(|K|, 3) = 1$. Then $\langle a\rangle$, $\langle b\rangle$ and $\langle c\rangle$ are all Sylow 3-subgroups of H. If a is conjugate in H to b^{-1}, we find $a \equiv b^{-1}$ (mod. K), $c = ab \in K$. Therefore, as $\langle a\rangle$ and $\langle b\rangle$ are conjugate in H, a and b are conjugate. Again, if a is conjugate to c, we find $c^2 = abc \equiv a^3 = 1$ (mod. K), another contradiction. Therefore a is conjugate to c^{-1}.

Let $C_1, \cdots, C_r$ be the conjugate classes of G containing elements from $P - Z$, and let $\tilde{C}_i = C_i \cap M$. Then Lemma 12 shows that $\tilde{C}_i$ is a class of M. We have seen in Section 3 that M contains t classes consisting of elements of $P - Z$, so $r = t$.

Denote by C_i also the sum of the elements of C_i, regarded as an element of the group ring of G (over the complex numbers). Let c_{ijk} be the coefficient of C_k in the product C_iC_j, and let s_{ijk} have the same significance for M and the $\tilde{C}_i$'s. If $c \in C_k$, then c_{ijk} is the number of solutions of $ab = c$, with $a \in C_i$, $b \in C_j$. Therefore Lemma 13 implies

$$c_{ijk} = s_{ijk}, \text{ unless perhaps } C_i = C_j = C_k^{-1} \tag{11}$$

(compare with Theorem 4.1 of [12]).

Recall that $\chi_i, \cdots, \chi_t$, $\theta_1, \cdots, \theta_s$ are all the irreducible characters of G, and that if $a \in P - Z$, $|C(a)| = p^{2n}$. Using (7), [2, formula (21)] yields

$$c_{ijk} = \frac{|G|}{p^{4n}}\left(\sum_1^t \frac{\chi_l(a)\chi_l(b)\chi_l(c^{-1})}{\chi_l(1)} + \sum_1^s \frac{c_m^3}{\theta_m(1)}\right) \tag{12}$$

where $a \in \tilde{C}_i$, $b \in \tilde{C}_j$, $c \in \tilde{C}_k$. The same formula for M yields, using (4)

$$\begin{aligned} s_{ijk} = \frac{p^{2n+1}q}{p^{4n}}\Bigg(&\sum_1^t \frac{\tilde{\zeta}_l(a)\tilde{\zeta}_l(b)\tilde{\zeta}_l(c^{-1})}{\tilde{\zeta}_l(1)} \\ &+ \sum_0^{q-1} \frac{\mu_l(a)\mu_l(b)\mu_l(c^{-1})}{\mu_l(1)} + \sum_1^{(p-1)q} \frac{\lambda_l(a)\lambda_l(b)\lambda_l(c^{-1})}{\lambda_l(1)}\Bigg) \\ = \frac{q}{p^{2n-1}}&\left(\frac{1}{q}\sum_1^t \tilde{\zeta}_l(a)\tilde{\zeta}_l(b)\tilde{\zeta}_l(c^{-1}) + q\right) \end{aligned}$$

From (6), (10) and the last equality we obtain

$$\begin{aligned} \sum_1^t \chi_l(a)\chi_l(b)\chi_l(c^{-1}) &= \varepsilon \sum_1^t \tilde{\zeta}_l(a)\tilde{\zeta}_l(b)\tilde{\zeta}_l(c^{-1}) \\ &= \varepsilon q\left(\frac{p^{2n-1}}{q}s_{ijk} - q\right) \end{aligned}$$

Substituting in (12) and using(11) yields

$$s_{ijk} = \frac{|G|}{p^{4n}}\left(\frac{1}{\chi_l(1)}(\varepsilon p^{2n-1}s_{ijk} - \varepsilon q^2) + \sum_1^s \frac{c_m^3}{\theta_m(1)}\right) \tag{13}$$

Suppose $\dfrac{\varepsilon|G|}{p^{2n+1}\chi_l(1)} = 1$. Then $\chi_l(1) = \dfrac{|G|}{p^{2n+1}}$,

which yields $\left(\dfrac{|G|}{p^{2n+1}}\right)^2 < |G|$, $|G| < p^{4n+2}$. The impossibility of this last inequality may be shown as follows: since P is a (TI) set, no non-identity element of P normalizes any Sylow p-subgroup of G, other than P. Therefore each Sylow p-subgroup different from P of G has exactly p^{2n+1} conjugates under P, and the total number of Sylow p-subgroups is $mp^{2n+1} + 1$, for some m, $m \neq 0$. As $M = N(P)$, this gives for the order of G: $|G| = p^{2n+1}q(mp^{2n+1} + 1) > p^{4n+2}$.

Therefore $\dfrac{|G|}{p^{2n+1}\chi_l(1)} \neq 1$. This means that the Equation (13) determines s_{ijk} uniquely, that is: *for all triples of indices* (i,j,k), *for which* $C_i = C_j = C_k^{-1}$ *is false,* s_{ijk} *is the same.* This is an assertion involving only the group M (not G), and it will be shown to be false, thus completing our proof.

Let $a \in P - Z$. Since P/Z is abelian, any conjugate of a in P belongs to aZ. As $a \notin Z$, there exist such conjugates different from a. Their number is a power of P. As $|aZ| = p$, this means that aZ consists of all the conjugates of a in P.

By (2), $t > 3$. Therefore, we can find an element $b \in P - Z$, which is conjugate neither to a nor to a^{-1}. Write $a = bc$. If $c \in Z$, then $a \in bZ$, so a is conjugate to b. Therefore $c \in P - Z$. Let i_1, j_1, k_1 be the indices for which $b \in C_{i_1}, c \in C_{j_1}, a \in C_{k_1}$, then $s_{i_1 j_1 k_1} \neq 0$. Moreover, for any $z \in Z$, we have $a = bz \cdot cz^{-1}$, and $bz \in C_{i_1}$, $cz^{-1} \in C_{j_1}$. Therefore $s_{i_1 j_1 k_1} \geqq p$. By the assertion made above, $s_{ijk_1} \geqq p$ for all pairs (i,j) excepting one pair at most. Hence, using (1) and (2),

$$\sum_{i,j} s_{ijk_1} \geqq (t^2 - 1)p > tqp = (p^{2n} - 1)p.$$

However, $\sum_{i,j} s_{ijk_1}$ is the total number of solutions of $xy = a$, with $x, y \in P - Z$. Each such pair (x, y) is determined by x alone, so the number of pairs is at most $|P - Z| = (p^{2n} - 1)p$. This is the desired contradiction.

5. *Extensions.* In Section 3, and the relevant parts of Section 4, we have not made use of the full assumption on G. Rather, the crucial point is the special structure of the group M, and even here we need slightly less than the fact that M is an S group. Also, the fact that q is prime is needed only to prove the inequalities $t > 3$, $(t^2 - 1) > tq$ and $t > q + 2$, all of which follow from (in fact, are equivalent to) the inequality $q < p^n - 1$. Thus we have proved

THEOREM 2. *There exists no simple group G, whose order is divisible by* 3, *and which satisfies the following conditions (where P denotes a Sylow* 3*-subgroup of G, $M = N(P)$ and $Z = Z(P)$).*

a. *P is an extra-special group, of order 3^{2n+1}, say, and of exponent* 3.

b. *P is a (TI) set.*

c. *$Z = Z(M)$, $M = N(Z)$, and M/Z is a Frobenius group with kernel P/Z.*

d. *If $q = |M:P|$, then $q < 3^n - 1$.*

Next, note that in the proof of Theorem 2 the fact that we are dealing with the prime 3 is needed in only one point, the proof of Lemma 13. Accordingly, let us introduce the following condition, where p is some given prime and G — a finite group.

(H) *If a, b, $c \in G$, $ab = c$, a, b, c have order p, and $c \in P$, where P is a Sylow p-subgroup of G, then, unless perhaps a, b, and c^{-1} are all conjugate, $a \in P$ and $b \in P$.* (This condition was introduced in [14]).

Then we have

THEOREM 3. *Theorem 2 still holds, if we change everywhere the prime 3 to the prime p and, in addition, assume that G satisfies (H).*

Now, consider what happens to Theorem 1 if we change 3 to p and add the condition (H). Most of the proof still holds, with the results of [13] and [14, Theorem 4] replacing those of [6] and [12]. The only difference is that we cannot rule out the possibilities that P is elementary abelian of order p^2 or extra-special of order p^3. However, this should not disturb us if we can show that P is, anyway, a (TI) set. For odd p this follows from

LEMMA 14. *Let p be an odd prime, and G a simple group whose order is divisible by p. Assume that each 2-maximal subgroup of G is p-nilpotent and that G satisfies (H). Then P, a Sylow p-subgroup of G, is a (TI) set.*

Proof. We know already that P is either elementary abelian or extra-special. Assume first that P is abelian. Then we can certainly assume $|P| > p$.

Let Q be another Sylow p-subgroup, and let $1 \neq a \in P \cap Q$. Let $b \in Q - P$. Then, a, b and ab all have order p. Thus (H) implies, since $b \notin P$, that b is conjugate to a. Changing a to a^{-1}, we find that b is also conjugate to a^{-1}, hence a and a^{-1} are conjugate, $a = g^{-1}a^{-1}g$, say. Then $g \in N(\langle a \rangle)$ but, as p is odd, $g \notin C(\langle a \rangle)$. Therefore, $N(\langle a \rangle)$ is not p-nilpotent, so, by Lemma 4, $N(\langle a \rangle)$ is conjugate to M. This, however, is impossible, M having no normal subgroups of order p.

Next, let P be extra-special, and Q and a as above. By Lemma 6, $a \notin Z$ and $Q \cap Z = 1$. Thus, if $1 \neq z \in Z$, it follows as for b above that z is conjugate to a. But then z is contained in a Sylow-intersection, again a contradiction.

Lemma 14 and our previous remarks yield

THEOREM 4. *Let p and G satisfy the same assumptions as in Lemma 14. Then* $G \cong PSL(2, q)$ *for some q.*

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University of Illinois,
Urbana